

Grothendieck ring class of Banana and Flower graphs

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Abstract

We define a special type of hypersurface varieties inside \mathbb{P}_k^{n-1} arising from connected planar graphs and then find their equivalence classes inside the Grothendieck ring of projective varieties. Then we find a characterization for graphs in order to define irreducible hypersurfaces in general.

1 Introduction

This is result from a short communication session from the summer school Applications of Algebra and Topology in Quantum Field Theory held in Villa de Leyva, Colombia during the summer of 2011.

The first three sections are mainly based on [1]. In the first section we introduce the concept of the Grothendieck ring of varieties together with some important results in this ring such as the inclusion-exclusion principle. In the second section we define a graph polynomial and a graph hypersurface and find some special properties about such polynomials. The third section is mainly concerned with the Grothendieck class of some special type of graphs, namely star, flower, polygon and banana graphs. In section four we explore graph hypersurfaces for graphs in general and find a necessary and sufficient condition for a graph to produce an irreducible graph hypersurface.

2 Grothendieck Ring of Varieties

The Grothendieck ring of varieties can be thought as a generalization of the Euler characteristic, [2]. It is the abelianization of the set group of symbols $[X]$ of isomorphism equivalence classes of algebraic varieties over a field k , [1, 2].

We start off with a characteristic zero field k and the the category of algebraic varieties \mathcal{V}_k defined over it. Then we look at the group $K_0(\mathcal{V}_k)$ of isomorphism classes $[X]$ of varieties X over the field k with the relation

$$[X] - [Y] = [X \setminus Y], \quad (1)$$

where $Y \subset X$ is a closed subvariety of X . This is called the *Grothendieck group of varieties* which then can be turned into a ring by defining the product of two

isomorphism classes by

$$[X] \cdot [Y] = [X \times Y]. \quad (2)$$

This can be also thought of as the quotient of the free abelian group generated by the symbols $[X]$ by the relation (1) and with the product (2), [2].

The use of the Grothendieck ring of varieties is very useful when looking for invariants. We call an *additive invariant* a map $\chi : \mathcal{V}_k \rightarrow R$, with values on a commutative ring R , that satisfies

1. $\chi(X) = \chi(Y)$ if X and Y are isomorphic
2. $\chi(X \setminus Y) = \chi(X) - \chi(Y)$, for $Y \subset X$ closed
3. $\chi(X \times Y) = \chi(X)\chi(Y)$

Thus, an additive invariant is the same as a ring homomorphism χ from the Grothendieck ring of varieties to the ring R , [1]. Some interesting additive invariants are the Euler characteristic, as mentioned before, and the Hodge polynomial, [1, 2].

For the following discussion, let $\mathbb{L} = [\mathbb{A}_k^1]$ be the equivalent class in the Grothendieck ring of varieties of the one dimensional affine space \mathbb{A}_k^1 . From the product (2) of $K_0(\mathcal{V}_k)$ we have that the Grothendieck class of the affine spaces \mathbb{A}_k^n are given in terms of \mathbb{L} as

$$[\mathbb{A}_k^n] = \mathbb{L}^n. \quad (3)$$

Also, let \mathbb{T} be the class of the multiplicative group k^\times , which is just \mathbb{A}_k^1 without a point. Thus we have that

$$\mathbb{T} = [\mathbb{A}_k^1] - [\mathbb{A}_k^0] = \mathbb{L} - 1, \quad (4)$$

where 1 is the class of a point in $K_0(\mathcal{V}_k)$.

\mathbb{L} and \mathbb{T} are useful to find the Grothendieck class of a variety together with the inclusion-exclusion principle

Theorem 2.1 (Inclusion-Exclusion Principle). *Let X, Y be varieties over k , then*

$$[X \cup Y] = [X] + [Y] - [X \cap Y] \quad (5)$$

Proof. Since we have that $X \setminus (X \cap Y) = (X \cup Y) \setminus Y$, then

$$[X] - [X \cap Y] = [X \setminus (X \cap Y)] = [(X \cup Y) \setminus Y] = [X \cup Y] - [Y], \quad (6)$$

and hence the result follows. \square

From this we find as a corollary the classes the projective spaces,

Corollary 2.2. *We have that*

$$[\mathbb{P}_k^n] = \frac{\mathbb{L}^{n+1} - 1}{\mathbb{L} - 1}, \quad (7)$$

where the fraction is taken as a short hand notation for the corresponding summation.

Proof. Since $\mathbb{P}_k^{n+1} \setminus \mathbb{A}_k^{n+1} \simeq \mathbb{P}_k^n$ for all $n \in \mathbb{N}$, by induction and the previous result, we have that

$$[\mathbb{P}_k^n] = 1 + \mathbb{L} + \mathbb{L}^2 + \cdots + \mathbb{L}^n, \quad (8)$$

from which the result follows. \square

3 Graph

3.1 Graph Hypersurfaces

In this section, let Γ be a connected planar graph with n edges and label them t_1, t_2, \dots, t_n . With this, define the graph polynomial associated to Γ by

$$\Psi_\Gamma(t) = \sum_{T \subset \Gamma} \prod_{e \notin E(T)} t_e, \quad (9)$$

where T is a spanning tree and $E(T)$ is the set of all edges of T . This polynomial is well define up to relabel of the edges.

For a graph with v vertices, the pigeon hole principle gives that all spanning trees have $v-1$ edges, as otherwise there would be a cycle. Hence the polynomial $\Psi_\Gamma(t)$ is a homogeneous polynomial of degree $n - v + 1$ and one can define the graph hypersurface associated to the graph Γ by

$$X_\Gamma = \{t = (t_1 : t_2 : \cdots : t_n) \in \mathbb{P}_k^{n-1} \mid \Psi_\Gamma(t) = 0\}. \quad (10)$$

Since $\Psi_\Gamma(t)$ is homogeneous, X_Γ is well define as a projective variety.

3.2 Dual Graph

Given a planar connected graph Γ , define the dual graph Γ^\vee by the following:

1. embed Γ in \mathbb{S}^2
2. for each region defined by Γ on \mathbb{S}^2 assign a vertex of Γ^\vee
3. if two regions share an edge, join the corresponding vertices on Γ^\vee

This definition depends on the particular embedding used. Different embeddings of the same graph Γ might lead to different dual graphs Γ^\vee , but the resulting graph polynomials $\Psi_{\Gamma^\vee}(t)$ are the same up to relabeling. This makes the graph hypersurface of the dual graph X_{Γ^\vee} to be a well defined object.

3.3 Cremona Transformation

The Cremona transformation on \mathbb{P}_k^{n-1} is given by

$$\mathcal{C} : \mathbb{P}_k^{n-1} \rightarrow \mathbb{P}_k^{n-1} \quad (11)$$

$$(t_1 : t_2 : \cdots : t_n) \mapsto \left(\frac{1}{t_1} : \frac{1}{t_2} : \cdots : \frac{1}{t_n} \right) \quad (12)$$

which is well defined outside the coordinate axes

$$\Sigma_n = \left\{ (t_1 : t_2 : \dots : t_n) \in \mathbb{P}_k^{n-1} \mid \prod_{i=1}^n t_i = 0 \right\}. \quad (13)$$

This Cremona transform is useful to relate the graph hypersurfaces of a graph Γ and its dual Γ^\vee . For this, consider the graph of the Cremona transform in $\mathbb{P}_k^{n-1} \times \mathbb{P}_k^{n-1}$ and define $\mathcal{G}(\mathcal{C})$ to be its closure. Then we have the two projections π_1 and π_2 of $\mathcal{G}(\mathcal{C})$ into $\mathbb{P}_k^{n-1} \times \mathbb{P}_k^{n-1}$ related by

$$\mathcal{C} \circ \pi_1 = \pi_2. \quad (14)$$

$\mathcal{G}(\mathcal{C})$ is then a closed subvariety of $\mathbb{P}_k^{n-1} \times \mathbb{P}_k^{n-1}$ with equations

$$t_1 t_{n+1} = t_2 t_{n+2} = \dots = t_n t_{2n}, \quad (15)$$

where we used the coordinates $(t_1 : t_2 : \dots : t_{2n})$ for $\mathbb{P}_k^{n-1} \times \mathbb{P}_k^{n-1}$.

Now, we have a relation between the graph polynomials of a graph Γ and its dual Γ^\vee given by

Proposition 3.1. *Let Γ be a graph with n edges. Then the graph polynomials of Γ and its dual Γ^\vee are related by*

$$\Psi_\Gamma(t) = \left(\prod_{i=1}^n t_i \right) \Psi_{\Gamma^\vee} \left(\frac{1}{t} \right), \quad (16)$$

and hence the corresponding graph hypersurfaces of Γ and Γ^\vee are related via the Cremona transformation by

$$\mathcal{C} (X_\Gamma \cap (\mathbb{P}_k^{n-1} \setminus \Sigma_n)) = X_{\Gamma^\vee} \cap (\mathbb{P}_k^{n-1} \setminus \Sigma_n) \quad (17)$$

This result also gives an isomorphism between the graph hypersurfaces of Γ and Γ^\vee away the coordinate axis which can be summarized in the following

Corollary 3.1. *Given a graph Γ , we have that the graph hypersurface of its dual is*

$$X_{\Gamma^\vee} = \pi_2(\pi_1^{-1}(X_\Gamma)), \quad (18)$$

where π_i are the projections given on (14). Also, we have that the Cremona transform gives a (birregular) isomorphism

$$\mathcal{C} : X_\Gamma \setminus \Sigma_n \rightarrow X_{\Gamma^\vee} \setminus \Sigma_n, \quad (19)$$

and the projection $\pi_2 : \mathcal{G}(\mathcal{C}) \rightarrow \mathbb{P}_k^{n-1}$ restricts to an isomorphism

$$\pi_2 : \pi_1^{-1}(X_\Gamma \setminus \Sigma_n) \rightarrow X_{\Gamma^\vee} \setminus \Sigma_n. \quad (20)$$

4 Banana and Flower graph hypersurfaces

4.1 Flower graphs

Let us start with the simplest of the graphs that define a graph hypersurface. Let Γ be a star graph with n edges. Then

$$\Psi_{\Gamma}(t) = 1, \quad (21)$$

and hence

$$X_{\Gamma} = \emptyset. \quad (22)$$

On the other hand, the dual graph Γ^{\vee} is a flower graph consisting with only one vertex and n loops. Therefore the graph polynomial associated to Γ^{\vee} is given by

$$\Psi_{\Gamma^{\vee}}(t) = t_1 t_2 \cdots t_n, \quad (23)$$

which makes its graph hypersurface to be

$$X_{\Gamma^{\vee}} = \{t \in \mathbb{P}_k^{n-1} \mid \Psi_{\Gamma^{\vee}}(t) = 0\} = \Sigma_n. \quad (24)$$

Therefore, by means of Corollary 3.1, we have an isomorphism between the graph hypersurfaces of the star graph and the flower graph away from the coordinate axis, which is indeed the case,

$$X_{\Gamma} \setminus \Sigma_n = \emptyset = X_{\Gamma^{\vee}} \setminus \Sigma_n. \quad (25)$$

It is straightforward to find the Grothendieck class of X_{Γ} to be

$$[X_{\Gamma}] = 0. \quad (26)$$

Now, for $X_{\Gamma^{\vee}}$, recall the class \mathbb{T}^{n-1} of the multiplicative group of \mathbb{P}_k^{n-1} . Notice that the multiplicative group is the complement of Σ_n on \mathbb{P}_k^{n-1} , which gives the following

Proposition 4.1. *The Grothendieck class of Σ_n is given by*

$$[X_{\Gamma^{\vee}}] = [\Sigma_n] = \frac{(1 + \mathbb{T})^n - 1 - \mathbb{T}^n}{\mathbb{T}} = \sum_{i=1}^{n-1} \binom{n}{i} \mathbb{T}^{n-1-i}. \quad (27)$$

Proof. Since $X_{\Gamma^{\vee}} = \Sigma_n = \mathbb{P}_k^{n-1} \setminus G_m$, where G_m is the multiplicative group of \mathbb{P}_k^{n-1} , the result follows from the definition of Grothendieck class, the relations given in (7) and (4). \square

4.2 Banana graphs

The next interesting type of graphs that give rise to graph hypersurfaces are the polygons. Let Γ be a polygon with n edges, i.e. a graph with n edges, n

vertices in which each vertex has degree 2. Here, the graph polynomial is then given by

$$\Psi_\Gamma(t) = t_1 + t_2 + \cdots + t_n, \quad (28)$$

and hence the graph hypersurface is the hyperplane

$$X_\Gamma = \{t \in \mathbb{P}_k^{n-1} | t_1 + t_2 + \cdots + t_n = 0\} =: \mathcal{L}. \quad (29)$$

Notice that choosing any $n-1$ points for t_1, t_2, \dots, t_{n-1} in \mathcal{L} gives a unique value for t_n , and hence we can identify \mathcal{L} with \mathbb{P}_k^{n-2} . Thus we find the Grothendieck class of \mathcal{L} to be given by

$$[\mathcal{L}] = [X_\Gamma] = [\mathbb{P}_k^{n-2}] = \frac{\mathbb{L}^{n-1} - 1}{\mathbb{L} - 1} = \frac{(\mathbb{T} + 1)^{n-1} - 1}{\mathbb{T}}. \quad (30)$$

The dual graph for Γ is a banana graph, which consists of 2 vertices of degree n each, and n edges joining them. The corresponding graph polynomial is then given by the $n-1$ symmetrical polynomial in t_i ,

$$\Psi_{\Gamma^\vee}(t) = t_2 t_3 \cdots t_n + t_1 t_3 \cdots t_n + \cdots + t_1 t_2 \cdots \hat{t}_i \cdots t_n + \cdots + t_1 t_2 \cdots t_{n-1}, \quad (31)$$

where \hat{t}_i means that the variable t_i is omitted from the product. In order to find its Grothendieck class, we first use the isomorphism between $X_\Gamma \setminus \Sigma_n$ and $X_{\Gamma^\vee} \setminus \Sigma_n$ to have

$$[X_\Gamma \setminus \Sigma_n] = [X_{\Gamma^\vee} \setminus \Sigma_n]. \quad (32)$$

From this, we can find the class of $X_\Gamma \setminus \Sigma$ by studying $\mathcal{L} \cap \Sigma_n$. First, we need a result that tells us how to find the Grothendieck class of a hyperplane section of a given class in the Grothendieck ring.

Lemma 4.1. *Let C be a class in the Grothendieck ring that can be written as a function of the torus class \mathbb{T} by means of a polynomial expression $C = g(\mathbb{T})$. Then the transformation*

$$\mathcal{H} : g(\mathbb{T}) \mapsto \frac{g(\mathbb{T} - g(-1))}{\mathbb{T} + 1} \quad (33)$$

gives an operation on the set of classes in the Grothendieck ring that are polynomial functions of the torus class \mathbb{T} that can be interpreted as taking a hyperplane section.

Since $[\Sigma_n] = \frac{(\mathbb{T}+1)^{n-1} - \mathbb{T}^n}{\mathbb{T}} = g(\mathbb{T})$, using the previous result gives that

$$[\mathcal{L} \cap \Sigma_n] = \frac{g(\mathbb{T}) - g(-1)}{\mathbb{T} + 1} = \frac{(1 + \mathbb{T})^{n-1} - 1}{\mathbb{T}} - \frac{\mathbb{T}^{n-1} - (-1)^{n-1}}{\mathbb{T} + 1}, \quad (34)$$

and with this, we find that

$$[X_{\Gamma^\vee} \setminus \Sigma_n] = [\mathcal{L} \setminus \Sigma_n] = [\mathcal{L}] - [\mathcal{L} \cap \Sigma_n] = \frac{\mathbb{T}^{n-1} - (-1)^{n-1}}{\mathbb{T} + 1}. \quad (35)$$

On the other hand, by definition, we have that the hypersurface X_{Γ^\vee} intersects the coordinate axis along the variety \mathcal{S}_n generated by the ideal

$$\mathcal{I} = (t_2 t_3 \cdots t_n, t_1 t_3 \cdots t_n, \dots, t_1 t_2 \cdots \hat{t}_i \cdots t_n, \dots, t_1 t_2 \cdots t_{n-1}). \quad (36)$$

With this, we can find the Grothendieck class of \mathcal{S}_n by

Lemma 4.2. *The class of \mathcal{S}_n is given by*

$$[\mathcal{S}_n] = [\Sigma_n] - n\mathbb{T}^n - 2 = \sum_{i=2}^{n-1} \binom{n}{i} \mathbb{T}^{n-1-i}. \quad (37)$$

Proof. Each coordinate hyperplane \mathbb{P}_k^{n-2} in Σ_n intersects the others along its own coordinate axis simplex Σ_{n-1} . Thus, to obtain the class of \mathcal{S}_n from the class of Σ_n , we just need to subtract the class of n complements of Σ_{n-1} in the n components of Σ_n , from which the result follows. \square

Finally, this results lead to the Grothendieck class of the graph hypersurface of the banana graph by

Theorem 4.3. *The class of the graph hypersurface of the banana graph is given by*

$$[X_{\Gamma^\vee}] = \frac{(\mathbb{T} + 1)^n - 1}{\mathbb{T}} - \frac{\mathbb{T}^n - (-1)^n}{\mathbb{T}} - n\mathbb{T}^{n-2}. \quad (38)$$

Proof. Writing $[X_{\Gamma^\vee}]$ as

$$[X_{\Gamma^\vee}] = [X_{\Gamma^\vee} \setminus \Sigma] + [X_{\Gamma^\vee} \cap \Sigma] = [X_{\Gamma^\vee} \setminus \Sigma] + [\mathcal{S}_n], \quad (39)$$

and using the previous results for $[X_{\Gamma^\vee} \setminus \Sigma]$ and $[\mathcal{S}_n]$ we find the desired result. \square

5 Irreducible Graph Hypersurfaces

It is important to notice that even though every planar connected graph give rise to a graph hypersurface, not every graph produces an irreducible graph hypersurface.

To see this, consider a planar connected graph Γ and define a *separating set* for Γ as a set of vertices in Γ whose removal will cause Γ to be disconnected. When we remove a vertex in Γ we also remove all the edges incident to it. Whenever there is a separating set consisting of only one vertex, we call this vertex a *separating vertex*.

With this, we can give a relation between separating sets and irreducibility of the graph hypersurfaces by the following result

Theorem 5.1. *If the graph Γ is not a tree and has a separating vertex, the corresponding graph hypersurface X_Γ is reducible.*

Proof. Let a be a separating vertex in Γ . Remove the vertex a and consider the disjoint components of the remaining graph. Since the components do not share an edge or a vertex, the choice of edges to remove in order to achieve a spanning tree on each component is independent of one another. Thus, grab each component independently and adjoin the vertex a together with the corresponding edges for that component. Then the corresponding graph polynomial for this component is a factor in the overall graph polynomial $\Psi_\Gamma(t)$. Since Γ is not a tree, at least one component give rise to a non trivial factor and hence X_Γ is reducible. \square

Now, suppose that the graph hypersurface X_Γ is reducible so that $\Psi_\Gamma(t)$ can be factored into non trivial factors

$$\Psi_\Gamma(t) = p(t)q(t). \quad (40)$$

As all the variables t_e have a power of either 0 or 1 in each monomial term in $\Psi_\Gamma(t)$, if t_e appears in $p(t)$, then it does not appear in $q(t)$ and vice-versa. Hence, the polynomials $p(t)$ and $q(t)$ separate the set of variables t_e appearing in $\Psi_\Gamma(t)$.

Therefore, each monomial in $\Psi_\Gamma(t)$ factors in a unique way as the product of monomials coming from $p(t)$ and $q(t)$. Hence the coefficients of $p(t)$ and $q(t)$ are all +1. Thus, both $p(t)$ and $q(t)$ can be thought of as being the graph polynomials of subgraphs of Γ .

Let P and Q be the graphs related with $p(t)$ and $q(t)$ respectively. Therefore $\Gamma/(P \cup Q)$ can only be the empty set or a tree, as if there was any cycles they will contribute to a factor to $\Psi_\Gamma(t)$.

With this discussion, we can generalize the previous result by means of defining a *separating tree* as a separating set whose edges on Γ form a tree. Hence, if we think of a vertex as a 0 edge tree, we can characterize irreducibility by the following result

Theorem 5.2. *If Γ is not a tree, Γ has a separating tree if and only if X_Γ is reducible.*

Also, it is important to notice that if a graph Γ produces a reducible graph hypersurface X_Γ so will its dual graph Γ^\vee .

Theorem 5.3. *X_Γ is irreducible if and only if X_{Γ^\vee} is irreducible.*

Proof. This result follows directly from Proposition 3.1. \square

References

- [1] Aluffi, P., Marcolli, M., *Feymann motives of banana graphs*, arXiv:0807.1690v2, (2008).
- [2] Sahasrabudhe, N., *Grothendieck Ring of Varieties*, Master Thesis, ALGANT Erasmus Mundus, Bordeaux, 2008.